COL7160 : Quantum Computing

Lecture 5: Qubit Copying and Teleportation

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1 Qubit Copying and the CNOT Gate

A natural question when first encountering quantum information is whether quantum states can be copied in the same way as classical bits. Classically, copying information is trivial: given a bit $b \in \{0, 1\}$, one can always produce two identical copies (b, b) without any restriction. Quantum mechanics, however, imposes fundamental limitations on this idea.

1.1 The CNOT Gate

The controlled-NOT (CNOT) gate is a two-qubit unitary operation defined by its action on computational basis states:

$$\text{CNOT} |a, b\rangle = |a, b \oplus a\rangle ,$$

where $a, b \in \{0, 1\}$ and $\oplus$ denotes addition modulo 2. In the ordered basis $\{|00\rangle , |01\rangle , |10\rangle , |11\rangle\}$, the matrix representation of CNOT is

$$\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} .$$

The first qubit is called the *control qubit*, and the second is called the *target qubit*. The target qubit is flipped if and only if the control qubit is in state $|1\rangle$.

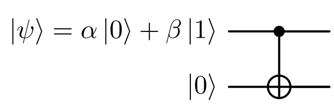
1.2 An Alternative Representation of the CNOT Gate

The CNOT gate can be interpreted as a controlled XOR operation acting on the computational basis labels. Its action is given by

$$\text{CNOT} |a\rangle |b\rangle = |a\rangle |b \oplus a\rangle , \quad a, b \in \{0, 1\},$$

where $\oplus$ denotes addition modulo 2.

This interpretation is conveniently represented by the following circuit diagram:



Consider the joint input state

$$|\psi\rangle \otimes |0\rangle = \alpha |00\rangle + \beta |10\rangle .$$

Applying the CNOT gate and using its basis-wise definition, we obtain

$$\alpha |00\rangle + \beta |10\rangle \xrightarrow{\text{CNOT}} \alpha |00\rangle + \beta |11\rangle .$$

It is important not to interpret this diagram as amplitudes "flowing" from the control qubit to the target qubit. The XOR operation acts on the classical labels of the computational basis states, determining how each basis vector is mapped. The amplitudes α and β remain unchanged and are merely reassigned to different basis states.

This perspective clarifies why the CNOT gate copies classical information (encoded in orthogonal states), but does not clone arbitrary quantum states. Instead, it typically produces entanglement between the control and target qubits.

1.3 Copying Classical Information

Suppose the target qubit is initialized in $|0\rangle$. Then,

$$\mathrm{CNOT}(|0\rangle\,|0\rangle) = |0\rangle\,|0\rangle\,, \qquad \mathrm{CNOT}(|1\rangle\,|0\rangle) = |1\rangle\,|1\rangle\,.$$

Thus, the value of the control qubit is reproduced on the target qubit. In this sense, CNOT acts as a copying gate for classical information encoded in orthogonal quantum states.
This is consistent with quantum mechanics because the states $|0\rangle$ and $|1\rangle$ are perfectly distinguishable (orthogonal).

1.4 Failure of Copying for Superpositions

Now consider an arbitrary qubit

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle\,, \qquad |\alpha|^2 + |\beta|^2 = 1.$$

Applying CNOT with the target initialized to $|0\rangle$ gives

$$\mathrm{CNOT}(|\psi\rangle\,|0\rangle) = \alpha\,|00\rangle + \beta\,|11\rangle\,.$$

This state is *entangled*. It cannot be written as $|\psi\rangle \otimes |\psi\rangle$. Hence, even though CNOT copies classical bits, it does not clone arbitrary quantum states.

2 The No-Cloning Theorem

The above observation leads to one of the most fundamental results in quantum information theory.

2.1 Statement of the Theorem

Theorem (No-Cloning Theorem). There does not exist a unitary operator U and a fixed state $|s\rangle$ such that

$$U(|\psi\rangle \otimes |s\rangle) = |\psi\rangle \otimes |\psi\rangle$$

for every quantum state $|\psi\rangle$.

2.2 Proof

We now prove that it is impossible to clone an arbitrary unknown quantum state using a unitary operation.
Assume, for the sake of contradiction, that there exists a unitary operator U such that for any quantum state $|\psi\rangle$,

$$U(|\psi\rangle \otimes |0\rangle) = |\psi\rangle \otimes |\psi\rangle\,,$$

where $|0\rangle$ is a fixed blank state. Let $|\psi_1\rangle$ and $|\psi_2\rangle$ be two arbitrary quantum states.
Then,

$$U(|\psi_1\rangle\,|0\rangle) = |\psi_1\rangle\,|\psi_1\rangle\,, \qquad U(|\psi_2\rangle\,|0\rangle) = |\psi_2\rangle\,|\psi_2\rangle\,.$$

Consider the inner product of the two output states:

$$\langle \psi_1 \mid \psi_2 \rangle^2 = \langle \psi_1 \otimes \psi_1 \mid \psi_2 \otimes \psi_2 \rangle.$$

Using properties of tensor products, this can be written as

$$\langle \psi_1 \otimes \psi_1 \mid \psi_2 \otimes \psi_2 \rangle = \langle \psi_1 \otimes 0 \mid U^\dagger U \mid \psi_2 \otimes 0 \rangle.$$

Since U is unitary, $U^\dagger U = I$. Therefore,

$$\langle \psi_1 \otimes 0 \mid U^\dagger U \mid \psi_2 \otimes 0 \rangle = \langle \psi_1 \otimes 0 \mid \psi_2 \otimes 0 \rangle.$$

Using the factorization of inner products over tensor products,

$$\langle \psi_1 \otimes 0 \mid \psi_2 \otimes 0 \rangle = \langle \psi_1 \mid \psi_2 \rangle \langle 0 \mid 0 \rangle.$$

Since $\langle 0 \mid 0 \rangle = 1$, we obtain

$$\langle \psi_1 \mid \psi_2 \rangle^2 = \langle \psi_1 \mid \psi_2 \rangle.$$

This equation implies that

$$\langle \psi_1 \mid \psi_2 \rangle \in \{0, 1\}.$$

Hence, $|\psi_1\rangle$ and $|\psi_2\rangle$ must be either orthogonal or identical. Therefore, no unitary operation can clone an arbitrary unknown quantum state.

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2.3 Consequences

The no-cloning theorem has profound implications:

- Quantum information cannot be backed up like classical data.
- Eavesdropping in quantum cryptography is detectable.
- Quantum teleportation cannot rely on copying; it must destroy the original state.

3 Entanglement and Bell States

3.1 Definition of Bell States

Bell states are maximally entangled two-qubit states defined as:

$$|\Phi^{\pm}\rangle = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}}, \qquad |\Psi^{\pm}\rangle = \frac{|01\rangle \pm |10\rangle}{\sqrt{2}}.$$

These four states form an orthonormal basis of the two-qubit Hilbert space.

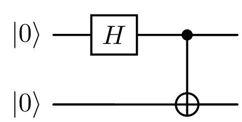
3.2 Preparation of All Four Bell States

Starting from the computational basis state $|00\rangle$, we now show how each of the four Bell states can be prepared using unitary operations.

(a) Preparation of $|\Phi^+\rangle$

Apply a Hadamard gate on the first qubit followed by a CNOT gate (with the first qubit as control and the second as target):

$$|00\rangle \xrightarrow{H \otimes I} \frac{|00\rangle + |10\rangle}{\sqrt{2}} \xrightarrow{\text{CNOT}} \frac{|00\rangle + |11\rangle}{\sqrt{2}} = |\Phi^+\rangle\,.$$

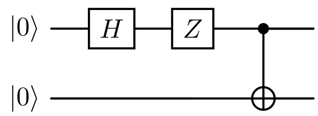


(b) Preparation of $|\Phi^-\rangle$

Starting from $|\Phi^+\rangle$, apply a Pauli-Z gate on the first qubit:

$$(Z \otimes I)\,|\Phi^+\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}} = |\Phi^-\rangle\,.$$

Equivalently, this can be achieved by inserting a Z gate after the Hadamard gate and before the CNOT gate.

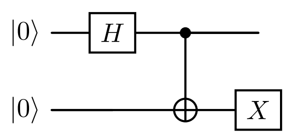


(c) Preparation of $|\Psi^+\rangle$

Starting from $|\Phi^+\rangle$, apply a Pauli-X gate on the second qubit:

$$(I \otimes X)\,|\Phi^+\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}} = |\Psi^+\rangle\,.$$

Thus, a circuit for $|\Psi^+\rangle$ is obtained by appending an X gate on the target qubit.

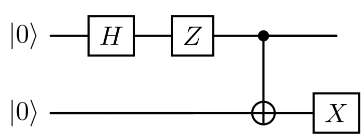


(d) Preparation of $|\Psi^-\rangle$

Finally, applying both a Z gate on the first qubit and an X gate on the second qubit to $|\Phi^+\rangle$ yields:

$$(Z \otimes X)\,|\Phi^+\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}} = |\Psi^-\rangle\,.$$

One possible circuit is:



Thus, all four Bell states can be prepared from $|00\rangle$ using only Hadamard, CNOT, and Pauli gates.

4 Quantum Teleportation

Quantum teleportation is a protocol that allows the transfer of an unknown quantum state from one party (Alice) to another (Bob), using shared entanglement and classical communication. No physical qubit carrying the state is sent, and the no-cloning theorem is not violated.

4.1 Initial Setup

Let Alice possess a qubit in an unknown state

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle\,,$$

which she wishes to teleport to Bob.

Alice and Bob also share a maximally entangled Bell state

$$|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}},$$

where Alice holds the first qubit of the Bell pair and Bob holds the second. This shared entangled state constitutes the prior shared information between Alice and Bob.
Thus, the total initial three-qubit state is

$$|\psi\rangle \otimes |\Phi^+\rangle = \frac{1}{\sqrt{2}}\Big(\alpha\,|000\rangle + \alpha\,|011\rangle + \beta\,|100\rangle + \beta\,|111\rangle\Big).$$

Here, the ordering of qubits is as follows:

- First qubit: Alice's unknown state $|\psi\rangle$
- Second qubit: Alice's half of the Bell pair
- Third qubit: Bob's half of the Bell pair

4.2 Applying the CNOT Gate

Alice now applies a CNOT gate on the first two qubits, with the first qubit as the control and the second qubit as the target.
Using the definition of the CNOT gate on computational basis states, we obtain

$$\frac{1}{\sqrt{2}}\Big(\alpha\,|000\rangle + \alpha\,|011\rangle + \beta\,|110\rangle + \beta\,|101\rangle\Big).$$

At this stage, the information contained in the unknown state $|\psi\rangle$ has begun to correlate with the shared entangled pair.

4.3 Applying the Hadamard Gate

Next, Alice applies a Hadamard gate to the first qubit. Recall that

$$H\,|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \qquad H\,|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}.$$

Applying this transformation to the state above yields

$$\frac{1}{2}\Big(\alpha(|000\rangle + |100\rangle) + \alpha(|011\rangle + |111\rangle) + \beta(|010\rangle - |110\rangle) + \beta(|001\rangle - |101\rangle)\Big).$$

This step is crucial: the Hadamard gate converts phase information into computational-basis information, allowing it to be extracted by measurement.

4.4 Rewriting the State

We now regroup the above expression according to the first two qubits:

$$\begin{aligned}\frac{1}{2}\Big(&\,|00\rangle\,(\alpha\,|0\rangle + \beta\,|1\rangle) + |01\rangle\,(\alpha\,|1\rangle + \beta\,|0\rangle)\\ &+ |10\rangle\,(\alpha\,|0\rangle - \beta\,|1\rangle) + |11\rangle\,(\alpha\,|1\rangle - \beta\,|0\rangle)\Big).\end{aligned}$$

This form makes the structure of the protocol explicit: depending on the measurement outcome of the first two qubits, Bob's qubit collapses to a state that is related to $|\psi\rangle$ by a known Pauli operation.

4.5 Measurement and Classical Communication

Alice now measures the first two qubits in the computational basis. The possible outcomes are $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$, each occurring with probability $1/4$.
After measurement:

- Alice's qubits collapse to the observed basis state.
- Bob's qubit collapses to a corresponding state shown above.

Alice then sends the two classical bits corresponding to her measurement outcome to Bob.

4.6 Bob's Correction

Based on the classical bits received from Alice, Bob applies the appropriate unitary correction to recover the original state $|\psi\rangle$:

Alice's outcome	Bob applies
00	I
01	X
10	Z
11	X then Z

After applying the correction, Bob's qubit is exactly in the state

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle .$$

4.7 Conceptual Remarks

- No quantum state is copied during the protocol; the original state is destroyed by Alice's measurement.
- Classical communication is essential; without it, Bob cannot recover $|\psi\rangle$.
- Entanglement acts as a resource that enables the transfer of quantum information.

Thus, quantum teleportation is consistent with the no-cloning theorem and illustrates the interplay between unitary evolution, entanglement, and measurement.

5 Notes on Dirac (Bra–Ket) Notation

In this section, we collect and clarify some commonly used notational conventions in Dirac notation, which are essential for correctly manipulating quantum states and operators.

5.1 Bras, Kets, and Inner Products

A *ket* $|\psi\rangle$ represents a vector in a complex Hilbert space, while a *bra* $\langle\psi|$ represents its Hermitian conjugate (dual vector). By definition,

$$\langle\psi| = (|\psi\rangle)^\dagger .$$

The inner product of two states $|a\rangle$ and $|b\rangle$ is written as

$$\langle a|b\rangle .$$

This expression is a complex scalar.
It is important to note that

$$\langle a|b\rangle \equiv \langle a| \, |b\rangle .$$

Writing $\langle a|b\rangle$ is simply a compact notation; the ket $|b\rangle$ is implicitly understood. Writing $\langle a| \, |b\rangle$ makes this explicit, but both expressions represent the same inner product.

5.2 What Is and Is Not Well-Defined

Only a bra followed by a ket defines an inner product. The following expressions are well-defined:

$$\langle a| \, |b\rangle \, , \qquad \langle a| \, U \, |b\rangle \, ,$$

where U is a linear operator.
A vertical bar | by itself has no meaning unless it is part of a bra or ket.

Remark (Tensor Product Notation). In practice, expressions such as $|a\rangle \, |b\rangle$ and $\langle a| \, \langle b|$ are often used informally to denote tensor products. More precisely,

$$|a\rangle \, |b\rangle \; \equiv \; |a\rangle \otimes |b\rangle \; = \; |ab\rangle \, ,$$

and

$$\langle a| \, \langle b| \; \equiv \; \langle a| \otimes \langle b| \; = \; \langle ab| \, .$$

Here, the order of states is important: $\langle a| \otimes \langle b|$ is the Hermitian adjoint of $|a\rangle \otimes |b\rangle$, and should not be confused with $\langle ba|$. While this shorthand is common, it is best interpreted explicitly as a tensor product to avoid ambiguity.

5.3 Matrix Interpretation

It is often helpful to interpret Dirac notation in terms of linear algebra. In this viewpoint:

- A ket $|\psi\rangle$ is a column vector.
- A bra $\langle\psi|$ is a row vector (the conjugate transpose of the ket).
- An operator U is a matrix.

Thus, the inner product

$$\langle\psi|\psi\rangle$$

corresponds to the matrix product

$$\psi^\dagger\psi,$$

which is a scalar. Similarly,

$$\langle\psi| \, U \, |\varphi\rangle$$

corresponds to $\psi^\dagger U \varphi$.
Thinking in terms of row vectors, column vectors, and matrices often helps avoid confusion when manipulating expressions in Dirac notation.

5.4 Adjoints and Operator Placement

A useful rule to remember is:

$$(A \, |v\rangle)^\dagger = \langle v| \, A^\dagger.$$

This rule explains why, when taking the adjoint of an expression, the order of operators is reversed and each operator is replaced by its Hermitian adjoint.
For example, if

$$|\psi_{\text{out}}\rangle = U \, |\psi_{\text{in}}\rangle \, ,$$

then

$$\langle\psi_{\text{out}}| = \langle\psi_{\text{in}}| \, U^\dagger.$$

5.5 Summary

- $\langle a|b\rangle$ and $\langle a| \, |b\rangle$ denote the same inner product.
- Only bra–ket expressions correspond to inner products.
- Interpreting bras and kets as row and column vectors is often the safest way to analyze expressions.
- Operator adjoints reverse order: $(AB)^\dagger = B^\dagger A^\dagger$.

Keeping these conventions in mind makes calculations involving quantum states systematic and less error-prone.